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Research article

New Banach Lattice Algebras and their Applications

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Abstract

Let X be a locally compact Hausdorf space and μ be a Radon measure on X. In this paper, we define a norm on the set $L^p(X,\mu)\cap L^\infty(X,\mu)$ to make it a Banach space, then we show that this space is a Banach algebra with pointwise multiplication. Moreover, we mention some applications of this space both as a Banach lattice and as a Banach algebra.

Keywords: Locally compact, Radon measure, Banach algebra, Banach lattice, Compact group, Haar measure, Ideals.

1. Introduction

Let X be a locally compact Hausdorf space and let μ be a Radon measure on X. Let $C_c(X,\mu)$ denote the space of all complex-valued continuous functions on X with compact support and $C_o(X,\mu)$ the space of all complex-valued continuous functions vanishing at infinity on X. For $1\le p<\infty$, $L^p(X,\mu)$ denote the space of equivalence classes of measurable functions f on X such that $\int_X |f(t)|^p\,dt<\infty$. This space is a Banach space normed by $\| f \|_p = \left(\int_X | f(t) |^p \, d\mu(t)\right)^{1/p}$. Let $L^\infty(X,\mu)$ be a space of equivalence classes of measurable functions f on X such that $ess\sup f < \infty$, where $ess\sup f := \inf\{\alpha > 0, | f(x) | < \alpha \; l.s. \; x \in X\}$, and by l.s. we mean locally almost.

This space is a Banach space with norm $\| f \|_\infty = ess\sup f$. For the sake of simplicity, we write $L^p(\mu)$ for $L^p(X,\mu)$ and $L^\infty(\mu)$ for $L^\infty(X,\mu)$. The spaces $L^p(\mu)$ and $L^\infty(\mu)$ are among the most important and the most elegant examples of Banach lattices. Here the lattice structure to be defined in the usual pointwise manner; see [6] for details.

Let $1 \le p < \infty$. We set $L^{p,\infty}(\mu) := L^p(\mu) \cap L^\infty(\mu)$. Our purpose in this paper is to study the space $L^{p,\infty}(\mu)$ as a Banach algebra. We also investigate some properties of this space as a Banach lattice.

2. $L^{p,\infty}(\mu)$ as a Banach Algebra

Let X be a locally compact Hausdorf space and μ be a Radon measure on X. Let $1 \le p < \infty$. In this section, we investigate the space $L^{p,\infty}(\mu)$ with norm $\| \cdot \|_{p,\infty} = \| \cdot \|_p + \| \cdot \|_\infty$. We first have the following result.

Lemma 2.1 *Let X be a locally compact Hausdorf space and μ be a Radon measure on X. For each $1 \le p < \infty$ the space $(L^{p,\infty}(\mu), \| . \|_{p,\infty})$ is a Banach space.*

Proof . Let (f_n) be a Cauchy sequence in $L^{p,\infty}(\mu)$. Then (f_n) is also Cauchy sequence in $L^p(\mu)$ and in $L^\infty(\mu)$. Then there exist $f \in L^p(\mu)$ and $g \in L^\infty(\mu)$ such that $\| f_n - f \|_p \to 0$ and $\| f_n - f \|_\infty \to 0$. Now, there exist a subsequence (f_{n_k}) of (f_n) such that $f_{n_k} \to f$ and $f_n \to g$ pointwise almost every where on X. Then $f = g$ almost every where on X and so $\| f_n - f \|_{p,\infty} \to 0$. Consequently, $L^{p,\infty}(\mu)$ is a Banach space.

Proposition 2.2 *Let X be a locally compact Hausdorf space and μ be a Radon measure on X and let $1 \le p < \infty$. Then the space $L^{p,\infty}(\mu)$ is a Banach algebra with pointwise multiplication. In particular $L^{p,\infty}(\mu)$ is an ideal in $L^\infty(\mu)$. Furthermore $L^{p,\infty}(\mu)$ is dense in $L^p(\mu)$.*

Proof. If $f, g \in L^\infty(\mu)$, then $\| f \cdot g \|_\infty \le \| f \|_\infty \| g \|_\infty$. Hence $f \cdot g \in L^\infty(\mu)$. On the other hand if $f \in L^p(\mu)$ and $g \in L^\infty(\mu)$ then

$$\| f \cdot g \|_p \le \| f \|_\infty \| g \|_p \, .$$

So, $f \cdot g \in L^p(\mu)$. Consequently, if $f, g \in L^{p,\infty}(\mu)$ then $f \cdot g \in L^{p,\infty}(\mu)$. Hence $L^{p,\infty}(\mu)$ is an algebra with pointwise multiplication. Let $f \in L^{p,\infty}(\mu)$ and $g \in L^\infty(\mu)$. Hence $f \in L^p(\mu)$ and $g \in L^\infty(\mu)$ then $f \cdot g, g \cdot f \in L^p(\mu)$ and from $f \in L^\infty(\mu)$ and $g \in L^\infty(\mu)$ we have $f \cdot g, g \cdot f \in L^\infty(\mu)$. Consequently

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$f \cdot g, g \cdot f \in L^{p,\infty}(\mu)$. Hence $L^{p,\infty}(\mu)$ is an ideal in $L^{\infty}(\mu)$. Finally, since $C_c(\mu) \subseteq L^{p,\infty}(\mu) \subseteq L^p(\mu)$ and $\overline{C_c(\mu)}^{\|\|_p} = L^p(\mu)$, it follows that $L^{p,\infty}(\mu)$ is dense in $L^p(\mu)$ and the proof is complete.

Proposition 2.3 *Let* X *be a locally compact Hausdorf space and* μ *be a Radon measure on* X. *Then the following statements are equivalent:*

(i). $L^{p,\infty}(\mu) = L^p(\mu)$,

(ii). X *is discrete.*

Proof. (i)$\Rightarrow$(ii) Suppose that $L^{p,\infty}(\mu) = L^p(\mu)$. Since $L^p(\mu) \subseteq L^{\infty}(\mu)$, consider the linear map $j: L^p(\mu) \to L^{\infty}(\mu)$. Then j is continuous; indeed, if (f_n) is a sequence in $L^p(\mu)$, $f_n \to f$ in $L^p(\mu)$ and $j(f_n) \to g$ in $L^{\infty}(\mu)$, then there exists a subsequence (f_{n_k}) of (f_n) such that $f_{n_k} \to f$ and $j(f_n) \to g$ pointwise almost every where on X. Consequently $j(f) = g$. Hence j is continuous by the closed graph theorem. Now, assume towards a contradiction that X is non-discrete. Then there exists sequence (V_n) of open subsets of X such that $\mu(V_n) \to 0$ as $n \to \infty$. Let $f_n = 1/\mu(V_n)^{1/p} \chi_{V_n}$ in $L^{p,\infty}(\mu)$ such that $\| f_n \|_p = 1$. then $\| f_n \|_{\infty} \to \infty$. This contradicts the continuity of j.

(ii)$\Rightarrow$(i) Let X be discrete. For each $f \in L^p(\mu)$, we obtain

$$\| f \|_{\infty} = \sup_{x \in X} | f(x) | < \sum_{x \in X} | f(x) |^p < \| f \|_p{}^p .$$

So, $f \in L^{\infty}$. Consequently $L^p(\mu) \subseteq L^{\infty}(\mu)$. Hence $L^{p,\infty}(\mu) = L^p(\mu)$.

The following result is an immediate consequence of Proposition 2.3.

Lemma 2.4 *Let* X *be a locally compact Hausdorf space and let* μ *be a Radon measure on* X *and* $1 < p < \infty$. *If* X *is discrete, then* $L^{p,\infty}(\mu)$ *is reflexive.*

Proposition 2.5 *Let* X *be a locally compact Hausdorf space and* μ *be a Radon measure on* X. *Then the following statements are equivalent:*

(i). $L^{p,\infty}(\mu) = L^{\infty}(\mu)$,

(ii). $\mu(X) < \infty$.

Proof. This follows from the fact that $L^{\infty}(\mu) \subseteq L^p(\mu)$ iff $\mu(X) < \infty$.

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In the following result, we characterize the dual space of $L^{p,\infty}(\mu)$; where $1\le p<\infty$ and μ is a Radon measure. This might be considered as another version of Riesz Representation Theorem.

Theorem 2.6 *Let* X *be a locally compact Hausdorf space and* μ *be a Radon measure on* X *and let* $1\le p<\infty$. *Then* $(L^{p,\infty}(\mu))^* = L^q(\mu)+(L^\infty(\mu))^*$ *with the action*

$$\langle \phi+\psi, f\rangle = \int_{\mathsf{X}} f\phi\, d\mu + \langle \psi, f\rangle.$$

The norm of $F=\phi+\psi$ is

$$\| F \|_{(L^{p,\infty})^*} = \inf \left\{ \max\{\|\phi\|_q, \|\psi\|_{\sup}\}, F=\phi+\psi, \phi\in L^q(\mu), \psi\in(L^\infty(\mu))^* \right\},$$

where $1/p+1/q=1$.

Proof. Let $F=\phi+\psi$ where $\phi\in L^q(\mu)$ and $\psi\in(L^\infty(\mu))^*$. For each $f\in L^{p,\infty}$ we have $|\langle F, f\rangle| = |\langle\phi, f\rangle+\langle\psi, f\rangle|$

$$\le |\langle\phi, f\rangle|+|\langle\psi, f\rangle|$$

$$\le \|\phi\|_q \| f\|_p + \|\psi\|_{\sup}\| f\|_\infty$$

$$\le \| f\|_{p,\infty} (\|\phi\|_q + \|\psi\|_{\sup})$$

$$\le \| f\|_{p,\infty} \max\{\|\phi\|_q, \|\psi\|_{\sup}\}.$$

So, we conclude that

$$\| F\|_{(L^{p,\infty})^*} \le \max\{\|\phi\|_q, \|\psi\|_{\sup}\}$$

and consequently

$$\| F \|_{(L^{p,\infty})^*} \le \inf \left\{ \max\{\|\phi\|_q, \|\psi\|_{\sup}\}, F=\phi+\psi, \phi\in L^q(\mu), \psi\in(L^\infty(\mu))^* \right\}.$$

Consider the Banach space $Y=L^p(\mu)\times L^\infty(\mu)$ with norm $\|(f,g)\| = \| f\|_p + \| g\|_\infty$. Thus $Y^* = L^q(\mu)\times(L^\infty(\mu))^*$ with norm $\|(\phi,\psi)\| = \max\{\|\phi\|_q, \|\psi\|_{\sup}\}$. Now, we define the map $I: L^{p,\infty}(\mu)\to Y$ by $I(f)=(f,f)$ for all $f\in L^{p,\infty}(\mu)$. Hence I is an isometry and $I(L^{p,\infty}(\mu))$ is closed in Y. The Hahn-Banach theorem extends F to a linear functional G on Y with $\| G\| = \| F\|_{(L^{p,\infty})^*}$. So, there are $\phi\in L^q(\mu)$ and $\psi\in(L^\infty(\mu))^*$ such that $F=\phi+\psi$ and

$$\| G \| = \max\{\| \phi \|_{\mathrm{q}}, \| \psi \|_{\sup}\} = \| F \|_{(L^{p,\infty})^*}.$$

For each $f \in L^{p,\infty}(\mu)$, we obtain

$$\langle G, I(f)\rangle = \int_X \phi f d\mu + \langle \psi, f\rangle = \langle F, f\rangle,$$

and so

$$\inf\{\max\{\| \phi \|_q, \| \psi \|_{\sup}\}, F = \phi + \psi, \phi \in L^q(\mu), \psi \in (L^\infty(\mu))^*\} \leq \| F \|_{(L^{p,\infty})^*},$$

as required.

3. $L^{p,\infty}(\mu)$ as a Banach Lattice

It is easy to check that for each $p \in [1,\infty)$ and each Radon measure μ the space $L^{p,\infty}(\mu)$ is a normed vector lattice under the usual partial order relation "$\leq$", defined by

$$f \leq g \iff f(x) \leq g(x) \quad \forall x \in \mathrm{X}.$$

Thus $L^{p,\infty}(\mu)$ is a Banach lattice and all properties of a Banach lattice remain valid for this Banach space. Since $L^{p,\infty}(\mu)$ is also an algebra, one of the most important properties of this Banach lattice which might be interesting is the structure of its lattice ideals. Recall that J is an ideal in a vector lattice V if for each $y \in V$ which $|y| \leq |x|$ for some $x \in \mathrm{J}$, we have $y \in \mathrm{J}$. This interesting structure can be found in the following lemma whose proof is presented here for the sake of completeness [1].

Lemma 3.1 *Let X be a Banach lattice as well as an algebra. If J is a closed ideal of X, then J is an algebraic ideal of X. Consequently, every closed lattice ideal in X is an algebraic ideal.*

Proof. Suppose J is a closed ideal of X. Let $g \in \mathrm{J}$ and $f \in X$. Since $|g| \in \mathrm{J}$, $\| f \| (|g|) \in \mathrm{J}$. Thus the relation

$$|fg| = |f| \cdot |g| \leq \| f \| \cdot |g|$$

implies that $fg \in \mathrm{J}$. Therefore; J is in fact an algebraic ideal of X.

4. Applications

On the one hand for each $p \in [1,\infty)$, the commutative Banach algebra (or Banach lattice) $L^p(\mathrm{X},\mu) \cap L^\infty(\mathrm{X},\mu)$, and, in particular the space of all essentially bounded integrable operators on X for the case p=1, is of interest in its own right. On the other hand, in the case where X is a locally compact group with left

Haar measure μ and p=1, the space $L^p(X,\mu)\cap L^\infty(X,\mu)$ is a Segal algebra with respect to its group algebra. The concept of an abstract Segal algebra was introduced and extensively studied by Burnham [3], and pursued by several authors as an improvement of the concept of a Segal algebra originated in the work of Reiter [11], see the recent works [2], [4], [5], [9].

Moreover, Lemma 3.1 shows that every closed lattice ideal in $L^{p,\infty}(\mu)$ is an algebraic ideal. This property might be useful for those who seek the existence of nontrivial invariant algebraic ideals under bounded linear operators on vector algebras. In fact any theorem that guarantees the existence of nontrivial invariant closed ideals under bounded linear operators on Banach lattices, also guarantees the existence of nontrivial invariant algebraic ideals under bounded linear operators on $L^{p,\infty}(\mu)$. Here are a few examples:

Example 1. An easy consequence of [[5], Proposition 2] is the following. Recall that an operator T on a vector lattice V is said to be positive if $Tv \geq 0$ whenever $v \geq 0$.“ *Every compact, quasinilpotent, positive operator on* $L^{p,\infty}(\mu)$ *has a nontrivial invariant algebraic ideal.”*

Example 2. The following is much stronger than the result stated in Proposition 3.2 and can be derived from [[1], Theorem 4.3]:

“ *Suppose* T *is a compact, quasinilpotent, positive operator on* $L^{p,\infty}(\mu)$. *Then every operator* S *on* $L^{p,\infty}(\mu)$ *that commutes with* T *has a nontrivial invariant algebraic ideal.”*

Example 3. Using [[2], Lemma 2.3.6], the compactness condition can be dropped in the statement of Proposition 3.3.

“ *Suppose* S *is a convex semigroup of quasinilpotent positive operators on* $L^{p,\infty}(\mu)$. *Then there exists a nontrivial algebraic ideal that is invariant under each member of* S.*”*

Remark 3.2 *Propositions 3.2 and 3.3 remain valid if we replace* $L^{p,\infty}(\mu)$ *by* $C(K)$, *where* K *is a compact Hausdorf space. In fact there are different kinds of semigroup of positive operators on* $C(K)$ *which possess a nontrivial algebraic ideal, see [3] and [4].*

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